PART II

METHOD OF MATHEMATICAL INDUCTION AND NEWTON BINOMIALS

This topic has two main contents:

First, we will learn about mathematical induction, an important and effective tool of mathematics. We will get acquainted and practice using this method to prove many different types of mathematical propositions.

Next, we will learn more deeply and completely about Newton's binomial formula and Pascal's triangle, as well as practice and apply them in solving problems.

After this topic, you can:

- Use mathematical induction to prove many different mathematical propositions.
- Use Newton's binomial formula and Pascal's triangle to expand expressions of the form $(a + b)^n$

LESSON 1. MATHEMATICAL INDUCTION METHOD

1. Mathematical induction method

By coloring on the square grid as shown below:

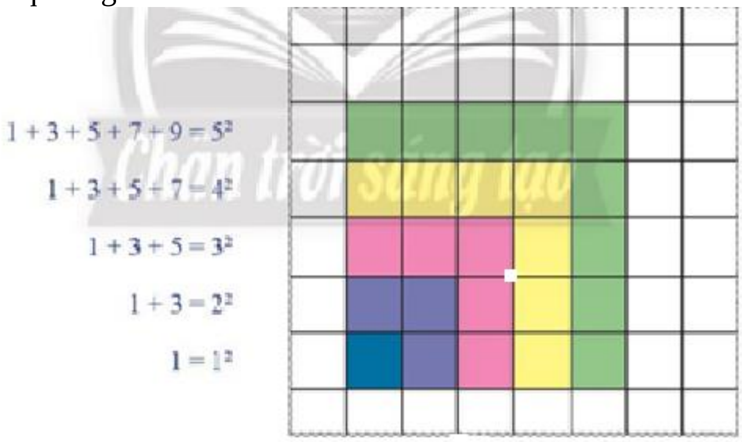

Figure 1

a student discovered the following formula:

$$1 + 3 + 5 + 7 + ... + (2n - 1) = n^2 \quad (1)$$

a) Show that formula (1) is correct for n = 1, 2, 3, 4, 5.
b) From coloring on the square grid as shown in Figure 1, the student affirms that formula (1) is definitely true for all natural numbers $n \geq 1$. Is such an assertion convincing? Why?

For each natural number $n \geq 1$, formula (1) is a mathematical proposition (proposition) since each of these propositions depends on the natural number $n \geq 1$.

Each time he filled in a row and added a column of squares, the student tested formula (1) with one more case of n. However, because the set N* is infinite, that method cannot prove that formula (1) is true for all $n \in N^*$. To achieve this, we need to use inference.

The principle of general induction gives us a powerful and effective method of inference to prove many propositions that depend on natural numbers.

Principle of mathematical induction

Suppose that for each natural number $n \geq 1$, P(n) is a proposition. Suppose the following two conditions are satisfied:
1) P(1) is correct,
2) For all natural numbers $k \geq 1$, if P(k) is true, then P(k+ 1) is true.
Then, P(n) is true for all natural numbers $n \geq 1$.

To prove that a statement dependent on natural numbers is true for all $n \in N^*$ using mathematical induction, we need to perform two steps:

Step 1. Point out the statement that is true for n = 1.

Step 2. The value is correct for the natural number $n = k \geq 1$ (called the inductive hypothesis), show that the statement is true for n = k + 1.

From there, according to the principle of mathematical induction, we conclude that the statement is true for all natural numbers $n \in N^*$?

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Example 1

Using the method of complete induction, prove that formula (1) is correct for n n∈ N*?

Solution

*Step 1***.** With n = 1, formula (1) becomes $1 = 1^2$.

This is a true statement, So (1) is true for n = 1.

Step 2. Suppose (1) is true for n = k≥ 1, which means we have

$1+ 3 +5 + 7 + ... + (2k - 1) = k^2$.

We need to prove (1) is true for n = k+ 1, which means we need to prove

$1+ 3 +5 + 7+ .... + (2k - 1) + (2k + 1) = (k + 1)^2$

According to the induction hypothesis, we have

l + 3 +5 + 7+.... (2k - 1) + (2k + 1) =[1 +3+ 5 +7 + (2k - 1)] + (2k + 1)

$= k^2 + (2k + 1)$

$= (k + 1)^2$.

So (1) is true for n = k+ 1.

According to the principle of mathematical induction, (1) is true for all n∈ N*.

<u>*Note*</u>: Sometimes, we need to prove that the statement P(n) is true for any natural number $n \geq n_0$, with n_0 being some natural number. Then, in the proof by mathematical induction method, in Step 1, we show that the statement is true for $n = n_0$, and in Step 2, we assume the statement is true for $n = k \geq n_0$.

Example 2

Prove that the inequality $2^n > n^2$ holds for all natural numbers n ≥5.

Solution

Step 1. With n = 5, we have 2^n = 25 = 32 and n^2 = 52 = 25. Because 32 > 25, the inequality is true for n = 5.

Step 2. Suppose the inequality holds for n = k ≥5, which means

$$2^k > k^2$$

We prove that the equality is true for n = k + 1, which means we need to prove

$$2^{k+1} > (k + 1)^2$$

Using the inductive assumption, with the note that k ≥ 5, we have

$$2^{k+1} = 2.\ 2^k$$
$$> 2k^2 = k^2 + k^2 \geq k^2 + 5k = k^2 + 2k + 3k$$
$$> k^2 + 2k + 1 = (k+1)^2$$

So the inequality is true for n= k + 1.

According to the principle of mathematical induction, the equality holds for all natural numbers n ≥ 5.

Exercises : a./ Prove that the following formula is true for all n ∈ N*

$$1 +2 + 3 +.....+n=\frac{n(n+1)}{2}$$

b./ Prove that the following inequality is true for all natural numbers n≥3.

$$2^{n+1} > n^2+n+2$$

2. Apply mathematical induction method

The method of mathematical induction is used in many different fields of mathematics (arithmetic, algebra, geometry, analysis, etc.). Below, we consider a few more applications.

Example 3

Prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in \mathbb{N}^*$.

Solution

For each $n \in \mathbb{N}^*$, consider the statement $(3^{2n+2} - 8n - 9) \vdots 64$. We need to prove that this statement is true for all $n \in \mathbb{N}^*$

Step 1. For n = 1, we have

$$3^{2n+2} - 8n - 9 = 3^4 - 8 - 9 = 81 - 8 - 9 = 64 \vdots 64.$$

So the statement is true for n = 1.

Step 2. Suppose the statement is true for $n = k \geq 1$, meaning there is $(3^{2k+2} - 8k - 9) \vdots 64$.
We need to prove that the statement is true for n = k+ 1, which means we need to prove

$$[3^{2(k+1)+2} - 8(k+1) - 9] \vdots 64$$

We have

$$3^{2(k+1)+2} - 8(k+1) - 9 = 9 \cdot 3^{2(k+1)} - 8k - 17 = 9(3^{2(k+1)} - 8k - 9) \vdots 64$$

This sum has the first term divisible by 64 (due to the induction assumption) and the second term is of course divisible by 64, so it is divisible by 64. So the statement is true for n = k+ 1.

According to the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}^*$.

Example 4

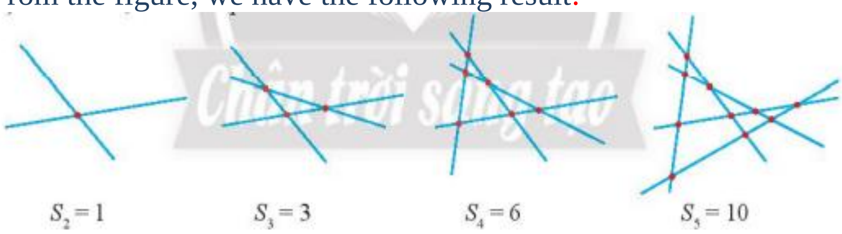
In a plane, give n ($n \geq 2$) straight lines, in which no two lines are parallel and no three lines are concurrent. Let Sn be the number of intersections of these n lines

a) Calculate S_2, S_3, S_4, S_5 corresponding to the case of 2, 3, 4,5 straight lines

b) From there, predict the formula for calculating S_n, and prove that formula using mathematical induction.

Solution

a) From the figure, we have the following result:



b) We have:
$S_2 = 1$,
$S_3 = 3 = S_2 + 2 = 1+ 2$;
$S_4 = 6 = S_3 +3 = 1+2+3$,
$S_5 = 10 = S_4+ 4 = 1+ 2+3 + 4$.

From there, we predict that

$$S_n = 1+ 2+3 +\ldots + (n - 1) = \frac{n(n-1)}{2} \qquad (2)$$

for all natural numbers $n \geq 2$.

We will prove this formula by mathematical induction,
Step 1. For n = 2, we have $S_2 = 1$ and $\frac{n(n-1)}{2}$ =1 so formula (2) is correct for n=2
Step 2. Suppose (2) is true for $n = k \geq 2$, meaning $S_k = \frac{k(k-1)}{2}$. We prove (2) is true for all n=k+ 1, meaning we need to prove:

$$S_{k+1} = \frac{(k+1)k}{2}$$

Call the (k + 1) th line d. According to the inductive assumption, k given lines intersect at $S_k = \frac{k(k-1)}{2}$ points . On the other hand, because no two lines are parallel and no three lines are concurrent, line d intersects those k lines at k different points and is different from the other S_k points. Therefore, the number of intersections of these (k + 1) lines is

$$S_{k+1}=S_k+k= \frac{k(k-1)}{2} +k = \frac{k^2-k+2k}{2} = \frac{(k+1)k}{2}$$

So formula (2) is correct for n = k + 1.
According to the principle of mathematical regularity, formula (2) is correct for all natural numbers $n \geq 2$.

Exercise:
1./ Prove that $n^3 + 2n$ is divisible by 3 for all $n \in N^*$.

2./ Prove that the following equality is true for all $n \in N^*$.

$$1+ q+q^2+q^3+q^4+\ldots\ldots q^{n-1}= \frac{1-q^n}{1-q}$$

3./ Prove that in the plane, n different straight lines passing through a point divide the plane into 2n parts ($n \in N^*$.)

4./ (Compound interest formula) An amount of money A VND (called capital) is deposited for a term at a bank according to the compound interest formula (interest after each period, if not withdrawn, will be added to the capital of the next period). Assuming the periodic interest rate r is constant across periods, the depositor does not withdraw capital and interest during the periods mentioned below. Let T_n be the total amount of capital and interest of the depositor after the n th term ($n \in N^*$).
a) Calculate T_1, T_2, T_3
b) From there, predict the formula for calculating Tn, and prove that formula using mathematical induction.

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EXERCISE

1./ Prove that the following formulas are true for all n ∈ N*.

a) 1. 2 + 2.3 + 3.4+ …….. = n(n+1)=$\frac{n(n+1)(n+2)}{3}$

b) 1 + 4 + 9 +…..n^2 = $\frac{n(n+1)(2n+!)}{6}$

c) 1 + 2 + 2^2 + 2^3 + 2^4 + …. 2^{n-1}= 2^n-1

2. Prove that, for all n ∈ N*.we have:

   a) 5^{2n} - 1 is divisible by 24;

   b) n^3 + 5n is divisible by 6

3. Prove that if x > - 1 then $(1 + x)^n \geq 1+ nx$ for all n ∈ N*.

4. Let a, b ≥0. Prove that the following inequality holds for all n ∈ N*.

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n$$

5. Prove that the following inequality holds for all natural numbers n≥ 2:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n} > \frac{2n}{n+1}$$

6. In a plane surface, let the polygon $A_1A_2A_3 \ldots A_n$, have n sides (n ≥ 3). Let S be the sum of the measures of the interior angles of the polygon.

   a) Calculate S_3,S_4,S_5, corresponding to the case of polygons being triangles, quadrilaterals, and pentagons.

   b) From there, predict the formula for calculating S_n, and prove the measurement formula by mathematical induction.

7. Every month, a person deposits a constant savings of one dong in the bank. Assume the monthly interest rate is constant r and follows the compound interest formula (previous month's interest is added to the next month's capital). Let T_n (n≥1) be the total capital and interest that person has in the bank at the time immediately after depositing the n + 1th account.

   a) Calculate T_1,T_2,T_3

   b) Predict the formula for calculating T_n and prove that formula using mathematical induction.

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LESSON 2. NEWTON BINOMIAL

I./ Newton's binomial formula

We have the expansion

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

or

$$(a + b)^3 = C_3^0 a^3 + C_3^1 a^2 b + C_3^2 ab^3 + C_3^3 b^3$$

In general, with any natural number n≥1, we have the expansion

$$(a + b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + \ldots + C_n^k a^{n-k} + \ldots + C_n^{n-1} ab^{n-1} + C_n^n b^n.$$

The rule or formula for expansion of $(a + b)^n$, where n is any positive integral power, is called binomial theorem. For any positive integral n

Note :

1) In the right of (1), the term $C_n^k a^{n-k} b^k$ (0≤k≤n) is called the *General Term* or (r + 1) th term. It is denoted by T_{k+1}.

$$T_{k+1} = C_n^k a^{n-k} b^k \quad (0 \le k \le n)$$

The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion.

2) There are (n + 1) terms in the expansion.
2. The 1st term is n and (n + 1) th term or the last term is b n
3. The exponent of 'a' decreases from n to zero.
4. The exponent of 'b' increases from zero to n.
5. The sum of the exponents of a and b in any term is equal to index n.

Example 1:
Expand $(x + 2)^6$ by binomial theorem.
Solution
Applying Newton's binomial formula, we have

$$(x + 2)^6 = C_6^0.x^6 + C_6^1.x^5.2 + C_6^2.x^4.2^2 + C_6^3.x^3.2^3 + C_6^4.x^2.2^4 + C_6^5.x.2^5 + C_6^6.2^6$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64.$$

Practice : Expand $(x+2)^6$ by binomial theorem   a) $(x - y)^6$;   b) $(1 + x)^7$.

Example 2: Expand (1.04)5 by the binomial formula and find its value to two decimal places.
Solution: Expand binomial theorem

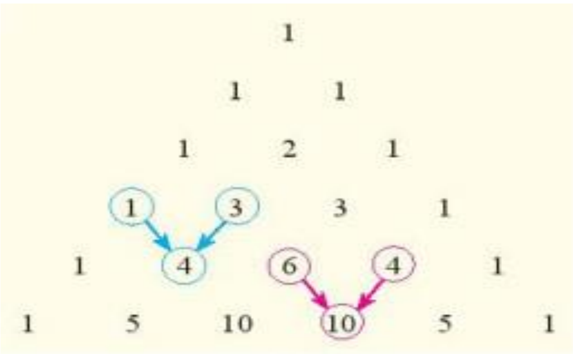
$$(1.04)^5 \quad = (1+0.04)^5$$
$$= 1. + 0.2 + 0.016 + 0.00064 + 0.000128 + 0.000\,000\,1024$$
$$= 1.22$$

II/. Pascal Triangle

From developed formulas

$$(a + b)^0 = 1;$$
$$(a + b)^1 = a + b;$$
$$(a + b)^2 = a^2 + 2ab + b^2;$$
$$(a + b)^3 = a^3 + 3a^2b + 3a\,b^2 + b^3;$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4;$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5;$$

the coefficients are written in tabulation as shown in Figure 2 below. If combinatorial notation is used, the table is obtained as shown in Figure 3.



Hình 2

Figure 2

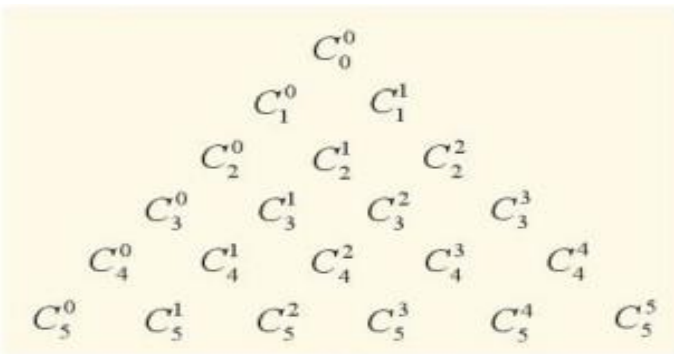


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Figure 3

From equalities such as

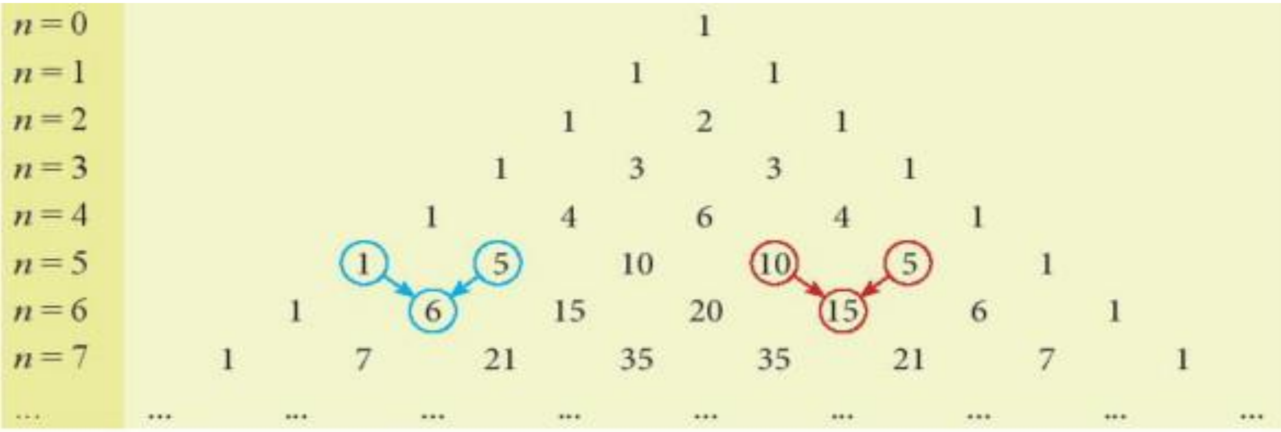
$C_3^0 = C_3^3 = 1$, $C_4^1 = C_4^3 = 4$,

$C_3^0 + C_3^1 = C_4^1$, $C_4^2 + C_4^3 = C_5^3$,

it is predictable that, for every $n \in N^*$

$C_n^k = C_n^{n-k}$ $\quad\quad\quad\quad$ $(0 \leq k \leq n)$.(2)
$C_n^{k-1} + C_n^k = C_{n+1}^k$ $\quad\quad\quad\quad$ $(1 \leq k \leq n)$. (3)

Using formula (3), note that $C_n^0 = C_n^m = 1$ for every $n \in N$, from the table in Figure 2, we can continue writing rows of numbers in turn to form a table of numbers, called a Pascal triangle (see Figure 4).



Hình 4

Figure 4

Example 3
Using the Pascal triangle, expand $(x - 1)^7$
Solution

$(x - 1)^7 = x^7 + 7x^6 . (-1) + 21x^5.(-1) + 35x^4.(-1)^3$
$+ 35.x^3 . (-1)^4 + 21.x^2.(-1)^5 + 7.x.(-1)^6 + (-1)^7$
$= 7x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$

Practice : Using the Pascal triangle, expand
a) $(2x + 1)^6$ $\quad\quad\quad\quad$ b) $(x - y)^7$

3. Application of the Newton's binomial formula

Example 4 : Find the coefficient of x^4y^6 in (2y – y)^{10}

Solution

Since

$$(2y-y)^{10} = C_{10}^{0}(2x)^{10} + C_{10}^{1}(2x)^{9}(-y) + \cdots + C_{10}^{k}(2x)^{10-k} + \cdots + C_{10}^{10}(-y)10$$

$$= 2^{10}C_{10}^{0}(x)^{10} - 2^{9}C_{10}^{1}x^{9}y+\ldots+(-1)^{k}2^{10-k}C_{10}^{k}x^{10-k}y^{k}+\ldots+C_{10}^{10}y^{10}$$

The term contains x^4y^6 corresponding to the value k = 6.

Therefore, the coefficient of x^4y^6 is $(-1)^6.2^4.C_{10}^{6} = 3360$

Example 5: Find the middle term of

$$\left(1-\frac{x^2}{2}\right)^{14}$$

Solution:

We have n = 14, then number of terms is 15.

So, 8th will be middle term.

$$T_8 = -\frac{429}{16}x^{14}$$

Example 6: Find the coefficient of x^{19} in

$$(2x^3 - 3x)^9$$

Solution

In the expansion of $(a + b)^n$, Here, a = $2x^3$, b = –3x, n = 9. First we find r.

Since

$$Tr + 1 = C_9^r 2^{9-r}(-3)^r x^{27-2r} \quad (1)$$

But we require x^{19}, so put

19 = 27 – 2r

2r = 8

r = 4

Putting the value of r in equation (1)

$$T5 = 1632960\ x^{19}$$

Hence the coefficient of x^{19} is 1632960.

Example 7: Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

Solution :
In the expansion of $(a + b)^n$
Let Tr + 1 be the term independent of x. We have a = 2x ^{2}, b = 1 x, n = 9

$$\text{Tr} + 1 = C_9^r 2^{9-r} x^{18-3r}$$

The power of x must be zero.
So 18 – 3r = 0 ⇒ r = 6
Therefore, Tr + 1 = 672 .

Example 8
Prove that the following equality í true for all n ∈ N*

$$C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n = 2^n$$

Solution
According to Newton's binomial formula, we have
$(x + 1)^n = C_n^0 (x)^n + C_n^1 (x)^{n-1} + C_n^2 (x)^{n-2} + \ldots C_n^{n-1} x + C_n^n$

Substituting x = 1 into the above formula, we get

$$C_n^0 + C_n^1 + C_n^2 + \ldots C_n^{n-1} x + C_n^n = 2^n$$

Example 9
Let the set A =$(a_1, a_2, \ldots a_n)$There are n elements. Does set A have multiple subsets?
Solution

Each subset of A has k (1 ≤ k ≤n) elements that are a convolutional combination k of A.
Thus, the number of such subsets is equal to C_n^k, On the other hand, there is a subset of 4 with no element (empty set), i.e. there C_n^0 = 1such subset.
Therefore, the number of subsets of A is equal to

$$C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n$$

According to Newton's binomial formula, we have
$C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^n = (1+1)^n = 2^n$

So set 4 has 2^n subsets.

Exercise

1. Expand the following by the binomial formula.

(i) $\left(x+\frac{1}{x}\right)^4$ (ii) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^5$ (iii) $\left(\frac{x}{2}-\frac{2}{y}\right)^4$

(iv) $(2x-y)^5$ (v) $\left(2a-\frac{x}{a}\right)^7$ (vi) $\left(\frac{x}{y}-\frac{y}{x}\right)^4$

(vii) $(-x+y^{-1})^4$

2. Compute to two decimal places of decimal by use of binomial formula.

(i) $(1.02)^4$ (ii) $(0.98)^6$ (iii) $(2.03)^5$

3. Find the value of

(i) $(x+y)^5+(x-y)^5$ (ii) $(x+\sqrt{2})^4+(x-\sqrt{2})^4$

4. Expanding the following in ascending powers of x

(i) $(1-x+x^2)^4$ (ii) $(2+x-x^2)^4$

5. Find

(i) the 5th term in the expansion of $\left(2x^2-\frac{3}{x}\right)^{10}$

(ii) the 6th term in the expansion of $\left(x^2+\frac{y}{2}\right)^{15}$

(iii) the 8th term in the expansion of $\left(\sqrt{x}+\frac{2}{\sqrt{x}}\right)^{12}$

(iv) the 7th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

6. Find the middle term of the following expansions

(i) $\left(3x^2+\frac{1}{2x}\right)^{10}$ (ii) $\left(\frac{a}{2}-\frac{b}{3}\right)^{11}$ (iii) $\left(2x+\frac{1}{x}\right)^7$

7. Find the specified term in the expansion of

(i) $\left(2x^2-\frac{3}{x}\right)^{10}$: term involving x^5

(ii) $\left(2x^2 - \frac{1}{2x}\right)^{10}$: term involving x^5

(iii) $\left(x^3 + \frac{1}{x}\right)^7$: term involving x^9

(iv) $\left(\frac{x}{2} - \frac{4}{x}\right)^8$: term involving x^2

(v) $\left(\frac{p^2}{2} + 6q^2\right)^{12}$: term involving q^8

8. Find the coefficient of

(i) x^5 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$

(ii) x^{20} in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{16}$

(iii) x^5 in the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{10}$

(iv) b^6 in the expansion of $\left(\frac{a^2}{2} + 2b^2\right)^{10}$

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